Board Game and Combinatorics of a Triangulated Square

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1. Introduction

There is a remarkable connection between Brouwer's fixed-point theorem and a board game called the Hex: the fact that there is no draw in the game is equivalent to the fixed-point theorem[1].

We propose a novel two-player board game on a triangulated square. We prove that there is no draw in the game, which can be considered as a generalization to the corresponding statement of the Hex game.

Keywords: combinatorics, board game, Hex game, Brouwer's fixed-point theorem, graph coloring

2. (Generalized) Hex game

Hex game | Two players take turns placing on the vertex orange and blue stones in a board composed of $n \times n$ hexagons. The orange player wins if his stones connect left-right, and the blue player wins if his stones connect up-down.

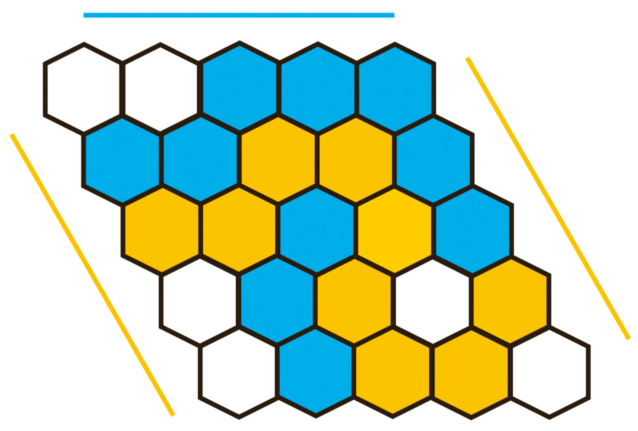

Figure 1. Hex game. Orange player wins.

Definition | Game board $\overset{\text{def}}{\Leftrightarrow}$ A square divided into triangles with n vertices inside. The two vertices on the diagonal of the square are painted orange and marked 0, and the two vertices on the other diagonal of the square are painted blue and marked n+1.

Generalized Hex game [2] | Two players take turns placing on the vertex orange and blue stones in a game board. The winner is the player who connects the diagonal line with stones of his color.

Hex theorem [2] | (generalized) Hex game doesn't result in a draw.

Proposition [2] | If the game board is *symmetric*, in generalized Hex game, first player always wins.

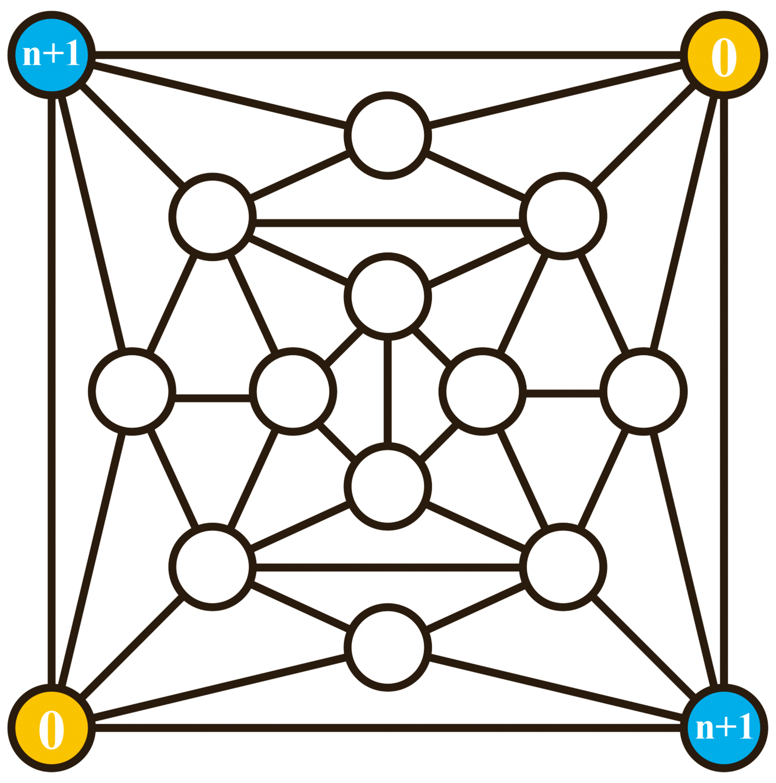

Figure 2. Game board

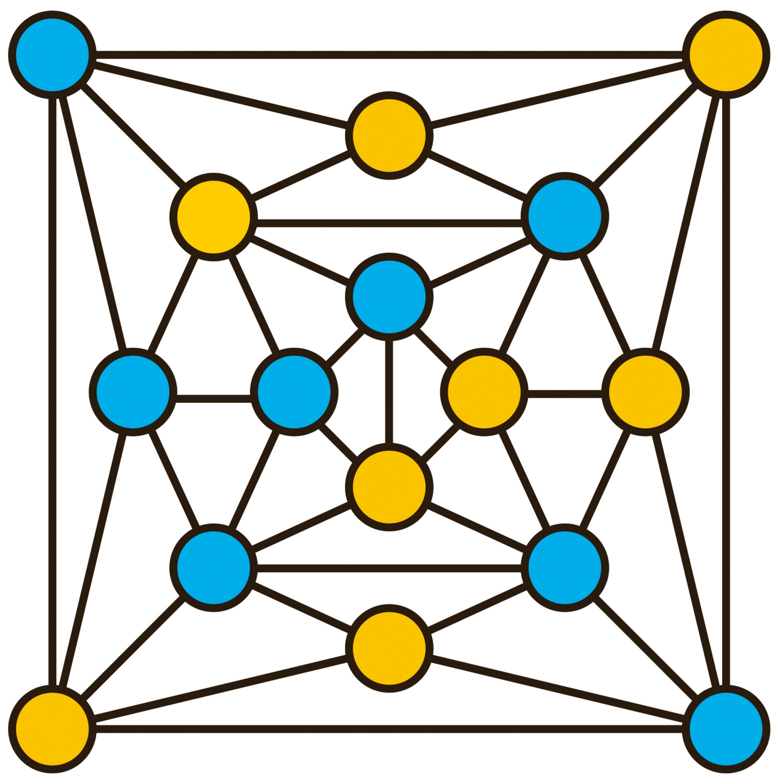

Figure 3. Generalized Hex game. Blue player wins.

3. Minmax game

Theorem 1 | On the game board, assign $1, \ldots, n$ different numbers to the n interior vertices. Then the following holds.

$$\min_{p \in \mathcal{P}_{0,0}} \max_{v \in p} v_{\text{num}} = \max_{p \in \mathcal{P}_{n+1,n+1}} \min_{v \in p} v_{\text{num}}$$

Where, $\mathcal{P}_{0,0}$ denotes the set of the paths connecting the two vertices 0 on the game board and v_{num} denotes the number assigned to vertex v.

Minmax game | On the game board, pieces numbered $1, \ldots, n$ (n: even) are divided into blue odd numbers $1, 3, \ldots, n-1$ and orange even numbers $2, 4, \ldots, n$, and two players take turns placing their pieces. When all pieces are placed, the player whose color satisfies "minmax = maxmin" is the winner.

Remark: Minmax game is an extension of generalized Hex game. Therefore, in minmax game, we can play generalized Hex game at the same time!

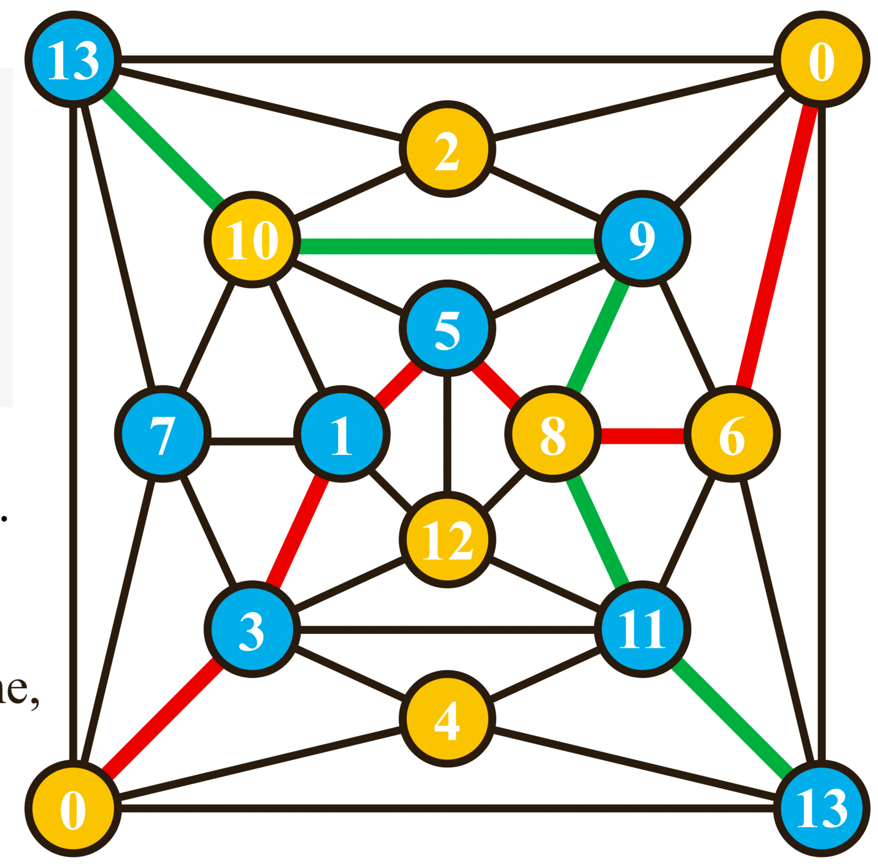

Figure 4. Example of minmax game.
Red line is a minmax path,
and green line is a maxmin path.
Orange player wins the minmax game, because the intersection 8 is orange.
But blue player wins the Hex game.

4. Combinatorial consequences

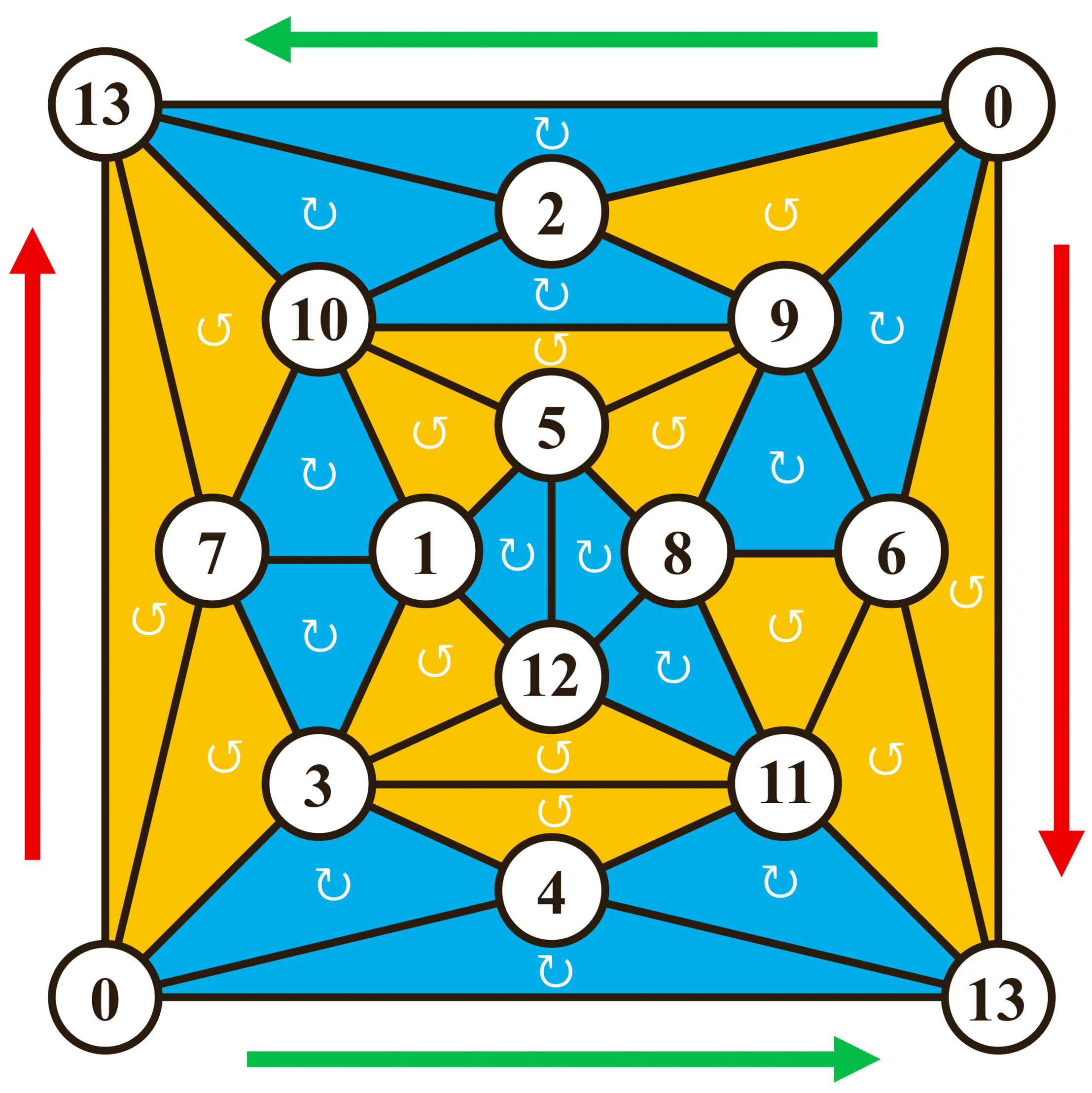

Figure 5. Face coloring by orientation

Notation | "C -" means "clockwise" and "CC -" means "counter-clockwise".

Theorem 2 | Let G be a triangulated planar graph. Number each vertex of G and face coloring by orientation. Then the following holds.

(1) #C -triangles − #CC -triangles
 = #C -boundary edges − #CC -boundary edges
(2) #C -connected components − #CC -connected components
 = #C -boundary triangles with CC -boundary edges
 −#CC -boundary triangles with C -boundary edges

5. Conclusion

In summary, the following relational equation holds.

Theorem 1 ⇔ **Hex theorem** ⇐ **Oriented Sperner's lemma [3]** $\overset{\text{Cor.}}{\Leftarrow}$ **Theorem 2 (1)**

References:
[1] David Gale, The Game of Hex and the Brouwer Fixed-Point Theorem, *The American Mathematical Monthly,* 1979, Vol.86, No.10, 818-827.
[2] Jiří Matoušek and Jaroslav Nešetřil, *Invitation to Discrete Mathematics*, pages 222-225, Oxford University Press, 2009.
[3] A. B. Brown and S. S. Cairns, *Proceedings of the National Academy of Sciences*, 1961, Vol.47, No.1, 113-114.